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SOLUTION OF THE SINGULARLY PERTURBED STABILITY PROBLEM*

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The singularly perturbed problem of the stability /1/ of solutions of a system of differential equations with derivatives multiplied by various powers of small parameters is considered. The effect of small parameters on the system dynamics is examined and the conditions under which the stability problem for the original system can be reduced to the analysis of a truncated system in special cases are determined. The general approach, based on separation of motions combined with methods of stability theory, enables the permissibility of idealizations /2/ in problems of mechanics to be examined (with a mathematically rigorous construction /3/ of simplified models as comparison systems for solving singularly perturbed problems) and enables estimates of the allowed parameter values to be obtained. The regular method used in this paper avoids a number of difficulties that arise in applied problems of mechanics, while generalizing existing results and producing a number of new results. The analysis focuses on problems of the dynamics of electromechanical systems in special critical cases.

1. Focusing on applications to problems of the dynamics of electromechanical systems (EMS), we will formulate the problem for an EMS modelling a gyrostabilization system /4/ in the critical case /1/. We know /5, 6/ that under appropriate assumptions the state of this EMS, characterized by a collection of n Lagrange and u Maxwell generalized coordinates, can be described by equations in general dynamic Maxwell-Lagrange (or Gaponov) form /7/. For the systems considered in this paper, the differential equations of perturbed motion can be represented in the form /8/ (we retain the notation of /8/, without the assumption of fast gyroscopes)

$$\frac{d}{dt}a\mathbf{q}_M^{\cdot} + (b+g)\mathbf{q}_M^{\cdot} = \mathbf{Q}_M' + \mathbf{Q}_{ME} + \mathbf{Q}_M'' \tag{1.1}$$

$$\frac{d}{dt}L\mathbf{q}_E^{\cdot} + R\mathbf{q}_E^{\cdot} = \mathbf{Q}_E' + \mathbf{Q}_{EM} + \mathbf{Q}_E'', \quad \frac{d\mathbf{q}_M}{dt} = \mathbf{q}_M^{\cdot}$$

$$\mathbf{q}_M = \|\mathbf{q}_1, \mathbf{q}_2\|^T, \quad \mathbf{Q}_M' = -e\mathbf{q}_M, \quad \mathbf{Q}_{ME} = A_M\mathbf{q}_E^{\cdot}, \quad A_M = \|0, A\|^T$$

$$\mathbf{Q}_{EM} = B_E\mathbf{q}_M^{\cdot}, \quad B_E = \|0, \ B\|, \ \mathbf{Q}_E' = -(\omega\mathbf{q}_1 + \Omega\mathbf{q}_E^{\cdot})$$

Here $\mathbf{q}_M$ is the n-dimensional vector of mechanical generalized (Lagrange) coordinates, $\mathbf{q}_1$ is the l-dimensional vector of mechanical control coordinates, $\mathbf{q}_E$ is the u-dimensional vector of electrical generalized (Maxwell) coordinates, $a = a(\mathbf{q}_M)$ and $b = b(\mathbf{q}_M)$ are $(n \times n)$ symmetric matrices of the positive definite quadratic form representing the kinetic energy of the mechanical part of the system and the positive semidefinite quadratic form in the expansion of the dissipative function of viscous friction forces, respectively, $g = g(\mathbf{q}_M)$ is the skew-symmetric matrix of gyroscopic coefficients, $L = \|L_{rs}\|$ and $R = R(\mathbf{q}_E^{\cdot})$ are $(u \times u)$ symmetric matrices of positive definite quadratic forms representing the electromagnetic energy of the system and the dissipative current function, respectively, L_{rs} are the coefficients of selfinduction and mutual induction of the windings in electrical circuits, $e = e(\mathbf{q}_M)$ is the $(n \times n)$ matrix of potential and non-potential forces, which depend on generalized coordinates, Q_{ME} and Q_{EM} are the mechanical generalized forces of electromagnetic origin (ponderomotive forces) and the electrical generalized forces of mechanical origin (counter-emf), $A = \|A_{kj}(L_{rs})\|$ is an $(l \times u)$ matrix, $B = \|B_{kj}(L_{rs})\|$ is a $(u \times l)$ matrix, $\mathbf{Q}_E'$ is the vector of electrical generalized forces corresponding to electrical generalized coordinates and $\mathbf{Q}_M''$ and $\mathbf{Q}_E''$ are sets of non-linear terms which depend on $\mathbf{q}_M$, $\mathbf{q}_M^{\cdot}$, $\mathbf{q}_E^{\cdot}$.

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System (1.1) is of order $(2n+u)$. A simplified model is often used in applications for analysing the dynamic properties of such systems (including their stability). In particular, assuming that the electrical circuits of servo-systems have a low inertia and treating the corresponding terms in the differential equations as small terms, we will ignore them and change to a truncated (idealized) model with inertialess electrical circuits /4, 9/ of the form

$$\frac{d}{dt} a\mathbf{q}_M^{\cdot} + (b+g)\mathbf{q}_M^{\cdot} = \mathbf{Q}_M' + \mathbf{Q}_{ME} + \mathbf{Q}_M'' \tag{1.2}$$
$$R\mathbf{q}_E^{\cdot} = \mathbf{Q}_E' + \overline{\mathbf{Q}}_E'', \quad \frac{d\mathbf{q}_M}{dt} = \mathbf{q}_M^{\cdot}$$

The order of this system is $2n$, i.e., lower than the order of the original system (1.1). We therefore need to rigorously justify the transformation to the truncated model in an infinite time interval (especially in stability problems). Note that the solution of such problems by various methods has been considered in many publications, including solutions for EMS of various types (see, e.g., /10, 11/), but the construction of model (1.2) as a comparison system for (1.1) is not considered in a rigorous framework in the literature. No conditions for the admissibility of such a system have been derived either.

A detailed analysis shows that these systems have singularities of the same kind as gyroscopic systems /8, 12/. This complicates the direct application of the results of singular perturbation theory. The EMS correspond to a class of singlarly perturbed systems /13/ whose differential equations can be represented as equations with the derivatives multiplied by various powers of small parameters. The motions in system (1.1) under the above assumptions are divided into three types of components with different time scales; the truncated sytsem (1.2) is not a limiting model for (1.1). Yet these problems can be very efficiently solved by the methods of Lyapunov stability theory /2, 14/.

2. In view of the above, we consider the stability problem as a singularly perturbed problem for systems of this type. Assume that the differential equations have been reduced to the form

$$\mu^{\alpha_i} d\varkappa_i/dt = \mathbf{K}\,(t,\,\mu,\,\varkappa)\ \ (i=1,2,3),\ \alpha_1=1,\ \alpha_2=2,\ \alpha_3=0 \tag{2.1}$$

where μ is a small positive parameter.

Assuming that in general the systems have a manifold of stationary states, we put

$$\varkappa_i = \mathbf{x}_i,\ \ \varkappa_3 = \|\mathbf{x}_3, \mathbf{z}\|^T,\ \ \mathbf{K}_i = P_i(\mu)\mathbf{x} + \mathbf{X}_i(t,\mu,\mathbf{z},\mathbf{x})\quad (i=1,\,2)$$
$$\mathbf{K}_3 = \|P_3(\mu)\mathbf{x} + \mathbf{X}_3(t,\mu,\mathbf{z},\mathbf{x}),\ \ \mathbf{Z}(t,\mu,\mathbf{z},\mathbf{x})\|^T,\ \ \mathbf{x} = \|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\|^T$$

where $\mathbf{x}_i$ are n_i-dimensional vectors of the main variables $(n=n_1+n_2+n_3)$ and z is the m-dimensional vector of critical /1/ variables. We assume that all functions in (2.1) are holomorphic in all the variables $\mathbf{z}$, $\mathbf{x}$ (in some domain), $\mathbf{X}_i$ and $\mathbf{Z}$ are non-linear vector functions of t, μ, $\mathbf{z}$ and $\mathbf{x}$ whose expansion does not contain terms of lower than the second degree with coefficients which depend on t and μ that vanish for $\mathbf{x}=0$ and any t, μ and $\mathbf{z}$, and $\mathbf{K}_3(t,\mu=0,\mathbf{z},\mathbf{x})$ does not contain terms with the fast variables $\mathbf{x}_2$.

System (2.1) is of the order of $(n+m)$. Consider the following problems: find the conditions when the solution of the problem of stability for system (2.1) can be reduced to the analysis of the stability of the truncated model; establish if the solution of the complete problem is close to the solution of the truncated system on an infinite time interval; find a technique for constructing an admissible truncated model for which reduction is valid in a dynamic setting. To solve these problem, we extend the method applied in /8, 15/ for special cases of systems of the form (2.1). Following /2/, we construct simplified models of various levels, introducing various approximate systems in accordance with our procedure.

As an approximate system for (2.1), we use the system linearized in μ (as is usually done in stability theory):

$$\mu\frac{d\mathbf{x}_1}{dt} = P_1{}^*\mathbf{x} + \mathbf{X}_1{}^*,\quad 0 = P_2{}^*\mathbf{x} + \mathbf{X}_2{}^* \tag{2.2}$$
$$\frac{d\mathbf{x}_3}{dt} = P_3{}^*\mathbf{x} + \mathbf{X}_3{}^*,\quad \frac{d\mathbf{z}}{dt} = \mathbf{Z}^*$$

This is a system of order $(n+m-n_2)$, and the asterisk denotes the retained terms with μ of not higher than first degree.

Then consider a system of zeroth order in μ as an approximate system for (2.1):

$$0 = P_i\mathbf{x} + \mathbf{X}_i\,(i=1,2),\ \ \frac{d\mathbf{x}_3}{dt} = P_3\mathbf{x} + \mathbf{X}_3,\ \ \frac{d\mathbf{z}}{dt} = \mathbf{Z} \tag{2.3}$$
$$P_i = P_i\,(\mu=0),\ \mathbf{X}_i = \mathbf{X}_i\ (t,\,\mu=0,\,\mathbf{z},\,\mathbf{x})\quad (i=1,\,2,\,3)$$

$$\mathbf{Z} = \mathbf{Z}\,(t, \mu = 0, \mathbf{z}, \mathbf{x})$$

This is the degenerate system of order $(n_3 + m)$ traditionally used in perturbation theory.

3. We solve the singularly perturbed problem of the stability of (2.1), (2.2). Here we have the critical case /1/: the characteristic equation of the first-approximation system for (2.1) has m zero roots. The other roots are determined from the equation

$$D\,(\lambda, \mu) = |\,M\,(\mu)\lambda - P\,(\mu)| = 0 \tag{3.1}$$

and $D\,(\lambda, \mu) = f_1\,(\lambda, \mu) + \mu^2 f_2\,(\mu, \mu) = 0$, where $f_1\,(\lambda, \mu)$ is a polynomial in λ which is obtained from $D\,(\lambda, \mu)$ when only terms linearized in μ are retained in each element of the determinant.

For system (2.2), the characterstic equation has the form $\lambda^m D_*\,(\lambda, \mu) = 0$, where $D_*\,(\lambda, \mu) = f_1\,(\lambda, \mu)$. The equation

$$D_*\,(\lambda, \mu) = 0 \tag{3.2}$$

is called the truncated equation, and the equations

$$D_1\,(\beta) = \begin{vmatrix} \beta E - P_{11} & -P_{12} \\ -P_{21} & -P_{22} \end{vmatrix} = 0 \tag{3.3}$$

$$D_2\,(\alpha) = |\,\alpha E - P_{22}\,| = 0 \tag{3.4}$$

are called auxiliary equations.

The methods of stability theory /1, 2/ lead to the following proposition.

Theorem 1. If for $|\,P\,(0)| \neq 0$, $|\,P_{22}\,(0)| \neq 0$ Eqs.(3.2) and (3.4) satisfy the Hurwitz conditions, then for sufficiently small μ the property of stability (asymptotic or non-asymptotic) of the zero solution of system (2.2) implies the corresponding property of stability of the zero solution of system (2.1).

For $m > 0$, any solution of system (2.1) of the form $\mathbf{z} = \mathbf{C}$, $\mathbf{x} = 0$ ($\|\mathbf{C}\|$ is sufficiently small) is also stable. The full system (2.1) has the holomorphic integral $\mathbf{z} + \mathbf{\Phi}\,(t, \mu, \mathbf{z}, \mathbf{x}) = \mathbf{A}$; the truncated system (2.2) has an integral of the form $\mathbf{z} + \boldsymbol{\varphi}\,(t, \mu, \mathbf{z}, \mathbf{x}_1, \mathbf{x}_3) = \mathbf{B}$, where $\mathbf{\Phi}$ and $\boldsymbol{\varphi}$ are holomorphic non-linear vector functions annihilating for $\mathbf{x} = 0$ and $\mathbf{x}_1 = 0$, $\mathbf{x}_3 = 0$ respectively, whose expansions contain no terms of lower than second degree in the variables $\mathbf{z}$, $\mathbf{x}$; $\mathbf{A}$ and $\mathbf{B}$ are arbitrary constant vectors, and

$$\mathbf{\Phi}\,(t, \mu, \mathbf{z}, \mathbf{x}) = \mathbf{\Phi}_0\,(t, \mathbf{z}, \mathbf{x}_3) + \mu \mathbf{\Phi}_1\,(t, \mu, \mathbf{z}, \mathbf{x})$$
$$\boldsymbol{\varphi}\,(t, \mu, z, \mathbf{x}_1, \mathbf{x}_3) = \boldsymbol{\varphi}_0\,(t, \mathbf{z}, \mathbf{x}_3) + \mu \boldsymbol{\varphi}_1\,(t, \mu, \mathbf{z}, \mathbf{x}_1, \mathbf{x}_3)$$

where $\mathbf{\Phi}_0$ and $\boldsymbol{\varphi}_0$ are identical on the solutions of the degenerate system.

Proof. Without going into details, we will highlight the main points of the proof (the proof is similar to that in /8/). For sufficiently small μ, under the condition of Theorem 1, the truncated system (2.2) can be represented in the form

$$\mu \frac{d\mathbf{x}_1}{dt} = \bar{P}_1{}^* \begin{Vmatrix} \mathbf{x}_1 \\ \mathbf{x}_3 \end{Vmatrix} + \bar{\mathbf{X}}_1{}^* \tag{3.5}$$
$$\frac{d\mathbf{x}_3}{dt} = \bar{P}_3{}^* \begin{Vmatrix} \mathbf{x}_1 \\ \mathbf{x}_3 \end{Vmatrix} + \bar{\mathbf{X}}_3{}^*, \quad \frac{d\mathbf{z}}{dt} = \bar{\mathbf{Z}}^*$$

This is a Lyapunov system /1/.

Denote by $\lambda = \lambda\,(\mu)$ the roots of Eq.(3.1) and by $\lambda_* = \lambda_*\,(\mu)$ the roots of Eq.(3.2). We can show that for $|\,P\,(0)\,| \neq 0$, $|\,P_{22}\,(0)| \neq 0$, n_3 roots λ and λ_* tend as $\mu \to 0$ to the values λ_0 of the roots of the degenerate equation

$$D_0\,(\lambda) = D\,(\lambda, 0) = 0 \tag{3.6}$$

and are equal to these roots in the limit; n_1 roots λ and λ_* may be represented in the form $\lambda\,(\mu) = \beta\,(\mu)/\mu$ and $\lambda_*\,(\mu) = \beta_*\,(\mu)/\mu$ respectively, where $\beta\,(\mu)$ and $\beta_*\,(\mu)$ tend as $\mu \to 0$ to the values β_0 of the roots of Eq.(3.3) and are equal to these roots in the limit. Estimating the errors $\Delta\lambda = \lambda\,(\mu) - \lambda_0$, $\Delta\lambda_* = \lambda_*\,(\mu) - \lambda_0$ for the roots of the first group and $\Delta\beta = \beta\,(\mu) - \beta_0$, $\Delta\beta_* = \beta_*\,(\mu) - \beta_0$ for the roots of the second group, we can show that for sufficiently small μ, $(n_1 + n_3)$ roots of Eq.(3.1) have negative real parts if the truncated Eq.(3.2) satisfies the Hurwitz conditions. The remaining n_2 roots of Eq.(3.1) for sufficiently small μ have negative real parts if Eq.(3.4) satisfies the Hurwitz conditions.

Hence we obtain that under the given conditions our systems for sufficiently small μ satisfy all the conditions of the corresponding Lyapunov theorems /1/, which proves Theorem 1.

Remarks. 3.1. This result defines the conditions when the singularly perturbed problem of the stability of systems (2.1), (2.2) has a solution (including the critical case $m > 0$). These conditions, in general, can be rewritten in a different form, noting that for sufficiently small μ all the roots of the truncated Eq.(3.2) lie in the left halfplane if the equations $D_0(\lambda) = 0$ and $D_1(\beta) = 0$ satisfy the Hurwitz conditions.

3.2. A similar result has been obtained for the singularly perturbed problem (2.1) and (2.3). We have determined the conditions when the truncated model corresponding to the limiting system is well posed.

3.3. The proposed method, combining the methods of stability theory and perturbation theory, can also be used to estimate the values μ for which the reduction to the truncated model is permissible. To this end, we need to consider the two subsystems into which the original perturbed system splits (in our setting): the subsystem corresponding to slow variables and the subsystem corresponding to fast variables. Following Chetayev /2/ and imposing conditions that ensure the stability of the solutions of the complete system (2.1) when the solutions of each of these subsystems are stable, we can obtain relationships for the allowed values of μ. Both the first and the second Lyapunov method may be used.

4. The problem of the stability of these systems is directly related to the problem of the closeness of the solutions of the complete and the truncated systems. The latter problem, as we known /2, 14/, is closely linked with Lyapunov theory and methods. Let us determine under what conditions the corresponding solutions of systems (2.1) and (2.2) are close in an infinite time interval. Denote by $\mathbf{x} = \mathbf{x}(t, \mu)$, $\mathbf{z} = \mathbf{z}(t, \mu)$ the solution of system (2.1) with the initial conditions $\mathbf{x}_0 = \mathbf{x}(t_0, \mu)$, $\mathbf{z}_0 = \mathbf{z}(t_0, \mu)$ and by $\mathbf{x}^* = \mathbf{x}^*(t, \mu)$, $\mathbf{z}^* = \mathbf{z}^*(t, \mu)$ the solution of the truncated system (2.2) with the initial conditions $\mathbf{x}_{i0}^* = \mathbf{x}_i^*(t_0, \mu)$ $(i = 1, 3)$, $\mathbf{z}_0^* = \mathbf{z}^*(t_0, \mu)$, and $\mathbf{x}_2^* = \mathbf{f}_2(t, \mu, \mathbf{z}^*, \mathbf{x}_1^*, \mathbf{x}_3^*)$, where $\mathbf{x}_2 = \mathbf{f}_2(t, \mu, \mathbf{z}, \mathbf{x}_1, \mathbf{x}_3)$ is the solution of the algebraic equation in system (2.2), $0 = P_2^*\mathbf{x} + \mathbf{X}_2^*$, for the variable $\mathbf{x}_2$.

The methods of stability theory prove the following theorem.

Theorem 2. If for $|P(0)| \neq 0$ Eqs.(3.3), (3.4) and (3.6) satisfy the Hurwitz conditions, then for sufficiently small μ for given numbers $\varepsilon > 0, \delta > 0, \gamma > 0$ (where ε and γ may be arbitrarily small) there exists μ_* such that the perturbed motion for $0 < \mu < \mu_*$ satisfies for all $t \geqslant t_0 + \gamma$ the inequalities $\|\mathbf{x} - \mathbf{x}^*\| < \varepsilon, \|\mathbf{z} - \mathbf{z}^*\| < \varepsilon$, if $\mathbf{x}_{i0} = \mathbf{x}_{i0}^*$ $(i = 1, 3)$, $\mathbf{z}_0 = \mathbf{z}_0^*$, $\|\mathbf{x}_{20} - \mathbf{x}_{20}^*\| < \delta$.

Without giving the detailed proof, we merely note that it involves investigating the solutions of systems (2.1) and (2.2) for sufficiently small μ, for which $\mathbf{x}_{i0} = \mathbf{x}_{i0}^*$ $(i = 1, 3)$, $\mathbf{z}_0 = \mathbf{z}_0^*$. Following Chetayev, we introduce the variables $\mathbf{a} = \mathbf{z} - \mathbf{z}^*$, $\mathbf{b}_i = \mathbf{x}_i - \mathbf{x}_i^*$ $(i = 1, 2, 3)$ and consider the differential equations for b_i that correspond to non-critical variables. These are obtained for Eqs.(2.1) and (2.2) and their integrals

$$\mu^i d\mathbf{b}_i/dt = \mathbf{B}_i(t, \mu, \mathbf{b}) \ (i = 1, 2), \ d\mathbf{b}_3/dt = \mathbf{B}_3(t, \mu, \mathbf{b}) \tag{4.1}$$

The behaviour of the variables $\mathbf{a}$, $\mathbf{b}$ is investigated in an infinite time interval, noting that $\mathbf{a}(t_0) = 0, \mathbf{b}_i(t_0) = 0$ $(i = 1, 3), \|\mathbf{b}_2(t_0)\| < \delta$, $\delta > 0$ is a given number.

Analysing system (4.1) and the structure of the integrals, we can show that under the given conditions these solutions have the following property: for given numbers $\varepsilon, \delta, \gamma$ (ε and γ may be arbitrarily small), there exists $\mu_* > 0$, such that for $0 < \mu < \mu_*$ for all $t \geqslant t_0 + \gamma$ we have $\|\mathbf{a}\| < \varepsilon, \|\mathbf{b}\| < \varepsilon$ with the given initial values. This proves the proposition.

Remarks. 4.1. A similar problem has been considered for system (2.1), (2.3).

4.2. Our analysis gives the conditions when the truncated model is permissible (in the problem of stability, in the problem of the dynamic characteristics of the transients). These studies are not only of theoretical interest (they generalize the results of singular perturbation theory to this critical case and this simplified system): they are also of interest in applications to problems of mechanics (in particular, to problems of EMS dynamics).

5. As an application of our results, let us consider the problem of stability for the EMS described in Sect.1 (in the critical case of m zero roots) as a singularly perturbed problem. We will construct the truncated model (as a comparison system) and determine the conditions when the reduction to the truncated model is permissible.

We denote as (5.1) the initial equations of motion represented in the form (1.1) (they are not written out in full here). Take $a = \|a_1, a_2\|^T$, $b = \|b_1, b_2\|^T$, $g = \|g_1, g_2\|^T$, $e = 0$, where a_i, b_i, g_i $(i = 1, 2)$ are appropriately dimensioned submatrices $(a_1, b_1, g_1$ are $(m \times n), m = n - l$, $m \geqslant l)$. In accordance with the proposed approach, treating the EMS as a singularly perturbed system, we reduce Eqs.(5.1) to the form (2.1). Assuming fast transients in the electric circuits of the servo-systems and putting

$$L_{rs}=L_{rs}{}^*\mu,\quad A_{kj}=A_{kj}{}^*\mu,\quad B_{kj}=B_{kj}{}^*\mu,\quad \tau=\mu t \tag{5.1}$$
$$\mathbf{x}_1=a\frac{d\mathbf{q}_M}{d\tau},\quad \mathbf{x}_2=L^*\mathbf{q}_E{}^\cdot,\quad \mathbf{x}_3=\dot{\mathbf{q}}_1,\quad \mathbf{z}=\mu a_1\frac{d\mathbf{q}_M}{d\tau}+(b_1{}^0+g_1{}^0)\mathbf{q}_M$$

where $\mu>0$ is a small parameter, we represent the equations in new variables in the form

$$\mu\frac{d\mathbf{x}_1}{d\tau}=-(\bar{b}+\bar{g})\mathbf{x}_1+\bar{A}^*\mathbf{x}_2+\mathbf{X}_1(\mu,\mathbf{z},\mathbf{x}) \tag{5.2}$$
$$\mu^2\frac{d\mathbf{x}_2}{d\tau}=-\bar{R}\mathbf{x}_2-\omega\mathbf{x}_3-\bar{\Omega}\mathbf{x}_2+\mu^2\bar{B}^*\mathbf{x}_1+\mathbf{X}_2(\mu,\mathbf{z},\mathbf{x})$$
$$\frac{d\mathbf{x}_3}{d\tau}=d_1\mathbf{x}_1,\quad \frac{d\mathbf{z}}{d\tau}=\mathbf{Z}(\mu,\mathbf{z},\mathbf{x}_1,\mathbf{x}_3)$$

In accordance with the previous results, we obtain that, for EMS with low-inertia electric circuits, two types of simplified models can be constructed, one corresponding to a truncated system linearized by μ (of the form (2.2)) and another corresponding to a limiting system in μ (of the form (2.3). Thus,

$$\frac{d}{dt}a\mathbf{q}_M{}^\cdot+(b+g)\mathbf{q}_M{}^\cdot=\mathbf{Q}_{ME}+\mathbf{Q}_M{}'' \tag{5.3}$$
$$R\mathbf{q}_E{}^\cdot=\mathbf{Q}_E{}'+\bar{\mathbf{Q}}_E{}'',\ \frac{d\mathbf{q}_M}{dt}=\mathbf{q}_M{}^\cdot$$
$$(b+g)\mathbf{q}_M{}^\cdot=\mathbf{Q}_{ME}+\bar{\mathbf{Q}}_M{}'' \tag{5.4}$$
$$R\mathbf{q}_E{}^\cdot=\mathbf{Q}_E{}'+\bar{\mathbf{Q}}_E{}'',\quad \frac{d\mathbf{q}_M}{dt}=\mathbf{q}_M{}^\cdot$$

of orders $2n$ and n, respectively. These approximate systems may be used as comparison systems (simplified models) for the original system of order $(2n+u)$ under appropriate conditions. In particular, it follows from our results that the transition to system (5.3) in the stability problem is permissible if μ is sufficiently small (the time constants of the electric circuits are sufficiently small) and the equations

$$|L\lambda+R^0+\Omega^0|=0,\quad |a^0\lambda+b^0+g^0|=0 \tag{5.5}$$
$$\begin{vmatrix} \multicolumn{2}{c}{b_1{}^0+g_1{}^0} & 0 \\ \multicolumn{2}{c}{(b_2{}^0+g_2{}^0)\lambda} & -A^0 \\ \omega^0 & 0 & R^0+\Omega^0 \end{vmatrix}$$

satisfy the Hurwitz conditions.

The stability property is then preserved and the solution of the complete system is close to the solution of the approximate system in an infinite time interval.

Remarks.5.1 Systems(5.3)(the linearized model) is identical with the well-known approximate model used for such EMS in /4, 9/. System (5.4), constructed here as a model that is limited (in μ), is a new approximate model, which is not traditional for problems of EMS dynamics and is fundamentally different from the known numerical models /9/ (we do not make any assumptions about the properties of the gyroscopes and the matrix g).

Example. Consider a uniaxial gyrostabilization system /4, 8/ modelled as an EMS with absolutely rigid elements, in which the stabilizing motor is a DC motor with independent excitation and with armature current control. We allow for the finite transient time in the electric circuits of the servo-systems.

The equations of perturbed motion may be represented /8/ in the form (1.1), where $n=2$, $u=3$:

$$\bar{A}\beta^{\cdot\cdot}+b_1\beta^\cdot-H\alpha^\cdot=\ldots,\quad J\alpha^{\cdot\cdot}+b_2\alpha^\cdot+H\beta^\cdot=-g_M i_2+\ldots \tag{5.6}$$
$$\sum_{j=1}^{3}L_{kj}i_j{}^\cdot+R_k i_k=E_k+\ldots\quad (k=1,2,3)$$
$$E_1=-\omega\beta,\ E_2=-\Omega i_1+g_E\alpha^\cdot,\ E_3=0$$

In accordance with the previous results, for systems with low-inertia electric circuits ($L_{kj}=L_{kj}{}^*\mu$, $k,j=1,2,3$; $g_M=g_M^*\mu$, $g_E=g_E{}^*\mu$) we may represent (5.6) as a singularly perturbed system of the form (5.2) and construct two types of simplified models. The model, linearized by μ, which is the old variables corresponds to a system of equations of the fourth order

$$A\beta^{\cdot\cdot} + b_1\beta^{\cdot} - H\alpha^{\cdot} = \ldots;\ J\alpha^{\cdot\cdot} + b_2\alpha^{\cdot} + H\beta^{\cdot} = -g_M i_2 + \ldots$$
$$R_1 i_1 = -\omega\beta + \ldots,\ R_2 i_2 = -\Omega i_1 + \ldots,\ R_3 i_3 = \ldots \tag{5.7}$$

corresponds to the well-known model /4/. The transition to (5.7) from the original system (5.6) of seventh order is permissible for sufficiently small μ, if in accordance with the conditions of Sect.5 $b_1 \neq 0$ (or $b_2 \neq 0$), $H\omega g_M\Omega > 0$, and the equation $|L\lambda + R + \Omega| = 0$ satisfies the Hurwitz conditions. Then the truncated model (5.7) is permissible in our sense if $\mu \leqslant \mu_*$. The estimate μ_* obtained here in line with Remark 3.3 following the ideas of /2/ and using relationships of the form $\sup_j |\mathrm{Re}\,\Delta\lambda_j| \leqslant \inf_j |\mathrm{Re}\,\Delta\lambda_{j0}|$ defines the region of allowed parameter values. In a special case ($b_1 = 0$, $L_{23}{}^* \neq 0$, all other $L_{kj}{}^* = 0\ (k \neq j)$), we obtain

$$g_M < \inf\left\{\frac{R_1R_2b_2H}{J\Omega\omega},\ \frac{R_1R_2H^3}{A\Omega\omega b_2},\ \frac{|L_{33}R_2 - L_{23}R_3|\,JR_2}{(L_{22}L_{33} - L_{23}{}^2)\,g_E}\right\}$$

Note that a simpler model is also valid in this case: it corresponds to the μ-limited system

$$b_1\beta^{\cdot} - H\alpha^{\cdot} = \ldots,\ b_2\alpha^{\cdot} + H\beta^{\cdot} = -g_M i_2 + \ldots$$
$$R_1 i_1 = -\omega\beta + \ldots,\ R_2 i_2 = -\Omega i_1 + \ldots,\ R_3 i_3 = \ldots$$

The transition to this second-order model is permissible when the inertia of the electric circuits in the EMS is sufficiently low and the corresponding conditions, similar to the above, are satisfied.

REFERENCES

1. LYAPUNOV A.M., The General Problem of the Stability of Motion, Collected Papers, 2, Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1956.
2. CHETAYEV N.G., On estimates of approximate integrations, PMM, 21, 3, 1957.
3. VORONOV A.A., An Introduction to the Dynamics of Complex Controlled Systems, Nauka, Moscow, 1985.
4. ISHLINSKII A.YU., Orientation, Gyroscopes, and Inertial Navigation, Nauka, Moscow, 1976.
5. MAXWELL J.C., Treatise on Electricity and Magnetism, in: Collected Papers on Electromagnetic Field Theory, Gostekhizdat, Moscow, 1954.
6. BULGAKOV B.V.,Oscillations, Gostekhizdat, Moscow, 1954.
7. NEIMARK YU.I. and FUFAYEV N.A., Dynamics of Non-holonomic Systems, Nauka, Moscow, 1967.
8. KUZ'MINA L.K., On the admissibility of simplified equations in the dynamics of gyroscopic systems, PMM, 52, 6, 1988.
9. RAUSHENBAKH B.V. and TOKAR E.N., Spacecraft Attitude Control, Nauka, Moscow, 1974.
10. KHODZHAYEV K.SH. and SHATALOV S.D., On the qualitative analysis of motion using asymptotic methods of non-linear mechanics, PMM, 44, 5, 1980.
11. MARTYNENKO YU.G., The Motion of a Rigid Body in Electric and Magnetic Fields, Nauka, Moscow, 1988.
12. MERKIN D.R., Gyroscopic Systems, Gostekhizdat, Moscow, 1956.
13. KUZ'MINA L.K., Methods of stability theory and singularly perturbed systems in mechanics, in: Proceedings of the 6th All-Union Conf. on Theoret. and Appl. Mechan., Tashkent, FAN, 1986.
14. GRADSHTEIN I.S., Solutions on the time halfline of differential equations with small parameters multiplying the derivatives, Matem. Sb., 32, 3, 1953.
15. KUZ'MINA L.K., On some properties of the solutions of singularly perturbed systems in one critical case, PMM, 46, 3, 1982.

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